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Fluid flow induced by a rotating disk of finite radius

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Abstract. The boundary-layer equations outside a rotating disk of radius a have been solved. It is shown that it is unnecessary to take special precautions for the sudden change in boundary conditions at the edge of the disk except if one is interested in the flow at distances which are smaller than about $10^{-3}a$ from the edge. The behaviour of the flow at large distances from the disk is investigated analytically with results which are confirmed by the numerical computations.

1. Introduction

The problem of solving the Navier–Stokes equations for a rotating disk of infinite radius has first been considered by von Karman [1]. Later, more accurate results have been obtained by Cochran [2], but since only the solution of a set of ordinary differential equations is involved, it is in the computer era no longer any problem to approximate the exact solution with arbitrary accuracy.

In the present paper the flow outside a circular rotating disk of finite radius a and zero thickness is considered on the basis of the boundary-layer equations. Since these equations are parabolic and the radial velocity component u is everywhere positive if the ambient fluid is at rest, there is to order Re^0 no feedback from the region $r > a$ toward the region $r < a$. The sudden change in the boundary conditions at $z = 0$ for $r = a$ from $u = 0$, $v = 0$ to $\partial u/\partial z = 0$, $\partial v/\partial z = 0$ (v = tangential velocity, z = axial coordinate) makes that there arises a double deck in the boundary layer of length $O(\mathrm{Re}^{-3/7})$ at both sides of the edge $r = a$ of the disk, see Smith [3]. For $r > a$ there is an inner and an outer Goldstein solution which differ numerically from the solutions behind the trailing edge of a flat plate [4] due to the different value of $\partial u/\partial z$ for $z = 0$ at the plate. In our case there are also Goldstein solutions for the tangential velocity v.

Special attention is given to the asymptotic behaviour of the velocity components u, v and w for $r \to \infty$. The components u and w decrease like r^{-1} and v like r^{-2}, provided z/r is kept constant. The boundary-layer thickness increases proportionally to r. Numerical solutions of the boundary-layer equations have been calculated for $r > a$, which for $r \to \infty$ are in complete agreement with the analytical results.

2. The boundary layer along the disk

For an axially-symmetric system the dimensionless equations of motion are (see e.g. [5]):

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \mathrm{Re}^{-1}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right\},$$

$$u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \mathrm{Re}^{-1}\left\{ \frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} \right\},$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \mathrm{Re}^{-1}\left\{ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right\},$$

while the equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{\partial w}{\partial z} = 0,$$

with w the axial velocity, p the pressure and ρ the fluid density. Lengths have been made dimensionless with a, velocities with Ωa and the pressure with $\rho\Omega^2 a^2$. The Reynolds number Re is $\Omega a^2/\nu$, with ν the kinematical viscosity coefficient. Boundary conditions are

$$z = 0: \quad u = 0, \quad v = r, \quad w = 0,$$
$$z \to \infty: \quad u \to 0, \quad v \to 0, \quad p \to 0.$$

Introduction of a stream function ψ by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

satisfies the continuity equation.

An exact solution can be obtained by putting

$$\begin{aligned} &\psi = \tfrac{1}{2}\,\mathrm{Re}^{-1/2} r^2 H(\mathrm{Re}^{1/2} z), \qquad p = \mathrm{Re}^{-1} P(\mathrm{Re}^{1/2} z), \\ &u = \tfrac{1}{2} r H'(\mathrm{Re}^{1/2} z), \qquad v = r G(\mathrm{Re}^{1/2} z), \qquad w = -\mathrm{Re}^{-1/2} H(\mathrm{Re}^{1/2} z). \end{aligned} \tag{2.1}$$

After substitution in the equations and in the boundary conditions, we obtain

$$\begin{aligned} &\tfrac{1}{2}H''' + \tfrac{1}{2}HH'' - \tfrac{1}{4}H'^2 + G^2 = 0, \\ &G'' + HG' - H'G = 0, \\ &\mathrm{Re}^{-1/2}(P' + H'' + HH') = 0, \\ &z = 0: \quad H(0) = 0, \quad H'(0) = 0, \quad G(0) = 1, \\ &z \to \infty: \quad H'(\infty) = 0, \quad G(\infty) = 0, \quad P(\infty) = 0. \end{aligned} \tag{2.2}$$

The numerical solution of boundary-value problem (2.2) can be obtained by a shooting procedure. It turns out that

$$\tfrac{1}{2}H''(0) = 0.510232619 \quad \text{and} \quad G'(0) = -0.615922014. \tag{2.3}$$

As shown by Zandbergen and Dijkstra [6] this non-linear problem has other solutions too, however, with doubtful physical significance. These will not be considered in this paper.

It is evident from the formulae (2.1) that $\partial p/\partial r$, $\partial^2 u/\partial r^2$, $\partial(u/r)/\partial r$, $\partial^2 v/\partial r^2$, $\partial(v/r)/\partial r$,

$\partial^2 w/\partial r^2$ and $r^{-1}\partial w/\partial r$ all are identically 0. Furthermore, $\partial p/\partial z = O(\mathrm{Re}^{-1/2})$. This implies that to $O(\mathrm{Re}^0)$ the solution of (2.2) is also the solution of the boundary-layer equations

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = \mathrm{Re}^{-1}\frac{\partial^2 u}{\partial z^2}, \qquad u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \mathrm{Re}^{-1}\frac{\partial^2 v}{\partial z^2}.$$

We transform all variables to $O(\mathrm{Re}^0)$ by introducing

$$\psi = \mathrm{Re}^{-1/2}\tilde{\psi}, \qquad w = \mathrm{Re}^{-1/2}\tilde{w}, \qquad z = \mathrm{Re}^{-1/2}\tilde{z},$$

which leads to

$$\begin{aligned} &u\frac{\partial u}{\partial r} + \tilde{w}\frac{\partial u}{\partial \tilde{z}} - \frac{v^2}{r} = \frac{\partial^2 u}{\partial \tilde{z}^2}, \qquad u\frac{\partial v}{\partial r} + \tilde{w}\frac{\partial v}{\partial \tilde{z}} + \frac{uv}{r} = \frac{\partial^2 v}{\partial \tilde{z}^2}, \\ &u = \frac{1}{r}\frac{\partial\tilde{\psi}}{\partial \tilde{z}}, \qquad \tilde{w} = -\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r}. \end{aligned} \tag{2.4}$$

These boundary-layer equations are parabolic with u everywhere positive. Hence, its $O(\mathrm{Re}^0)$ solution is not influenced by the region $r > 1$. This result holds only for zero angular velocity of the ambient fluid since otherwise there will be regions with negative u, see [6].

3. The inner Goldstein solution

At the edge of the disk there is a sudden change in the boundary conditions for $\tilde{z} = 0$:

$$u = 0 \text{ is changed into } \frac{\partial u}{\partial \tilde{z}} = 0 \text{ for } r > 1,$$

$$v = 1 \text{ is changed into } \frac{\partial v}{\partial \tilde{z}} = 0 \text{ for } r > 1.$$

For $r \leq 1$ we have $\partial u/\partial \tilde{z} = \frac{1}{2}H''(0)$ and $\partial v/\partial \tilde{z} = G'(0)$ of which the numerical values are given by (2.3). Hence, near $r = 1^+$ and $z = 0$ there must be a small region where $\partial u/\partial \tilde{z}$ increases from 0 to $\frac{1}{2}H''(0)$ and $\partial v/\partial \tilde{z}$ decreases from 0 to the negative value of $G'(0)$. This region will be determined by $r = 1 + x$ with x small, while we assume its extension in $\tilde{z}$-direction to be $O(x^\beta)$ with $\beta > 0$. The stream function $\tilde{\psi}(x, \tilde{z})$ will be $O(x^\alpha)$, which means $u = O(x^{\alpha-\beta})$, $w = O(x^{\alpha-1})$. Finally, v is assumed to be $1 + O(x^\gamma)$.

The various terms in the boundary-layer equations (2.4) then have the following orders of magnitude:

$$u\frac{\partial u}{\partial r} + \tilde{w}\frac{\partial u}{\partial \tilde{z}} = O(x^{2\alpha-2\beta-1}), \qquad \frac{v^2}{r} = O(x^0), \qquad \frac{\partial^2 u}{\partial \tilde{z}^2} = O(x^{\alpha-3\beta}),$$

$$u\frac{\partial v}{\partial r} + \tilde{w}\frac{\partial v}{\partial \tilde{z}} = O(x^{\alpha-\beta+\gamma-1}), \qquad \frac{uv}{r} = O(x^{\alpha-\beta}), \qquad \frac{\partial^2 v}{\partial \tilde{z}^2} = O(x^{\gamma-2\beta}).$$

Furthermore, $\partial u/\partial \tilde{z} = O(x^{\alpha-2\beta})$ and $\partial v/\partial \tilde{z} = O(x^{\gamma-\beta})$ must both be of order $O(x^0)$, due to their finite value. This leads to $\alpha = 2\beta$ and $\gamma = \beta$. Since in each equation there must always be two terms of largest magnitude, both equations yield

$$2\beta - 1 = -\beta \Rightarrow \beta = \tfrac{1}{3}\,, \qquad \alpha = \tfrac{2}{3} \quad \text{and} \quad \gamma = \tfrac{1}{3}\,.$$

In first approximation, the terms v^2/r and uv/r are clearly unimportant.

Suitable new coordinates in the region of the inner Goldstein solution are

$$\xi = x^{\frac{1}{3}} = (r-1)^{\frac{1}{3}} \quad \text{and} \quad \eta = \tilde{z}/x^{\frac{1}{3}} = \tilde{z}/\xi\,. \tag{3.1}$$

The equations of motion in the new coordinates are

$$\begin{aligned} &u\xi\,\frac{\partial u}{\partial \xi} + (3\xi^2\tilde{w} - \eta u)\,\frac{\partial u}{\partial \eta} - \frac{3\xi^3}{1+\xi^3}\,v^2 = 3\xi\,\frac{\partial^2 u}{\partial \eta^2}\,, \\ &u\xi\,\frac{\partial v}{\partial \xi} + (3\xi^2\tilde{w} - \eta u)\,\frac{\partial v}{\partial \eta} + \frac{3\xi^3}{1+\xi^3}\,uv = 3\xi\,\frac{\partial^2 v}{\partial \eta^2}\,. \end{aligned} \tag{3.2}$$

Expansions for $\xi \to 0$ then become

$$\begin{aligned} &\tilde{\psi} = \xi^2 f_0(\eta) + \xi^3 f_1(\eta) + O(\xi^4)\,, \\ &u = \frac{1}{r}\,\frac{\partial\tilde{\psi}}{\partial \tilde{z}} = \xi f_0'(\eta) + \xi^2 f_1'(\eta) + O(\xi^3)\,, \\ &v = 1 + \xi g_0(\eta) + O(\xi^2)\,, \\ &\tilde{w} = -\frac{1}{r}\,\frac{\partial\tilde{\psi}}{\partial r} = -\xi^{-1}\{\tfrac{2}{3} f_0(\eta) - \tfrac{1}{3}\eta f_0'(\eta)\} - \{f_1(\eta) - \tfrac{1}{3}\eta f_1'(\eta)\} + O(\xi)\,. \end{aligned} \tag{3.3}$$

The functions f_0, f_1 and g_0 are determined by differential equations obtained by substitution of the expansions (3.3) into equations (3.2). This yields

$$\begin{aligned} &3f_0''' + 2f_0 f_0'' - f_0'^2 = 0\,, \\ &3g_0'' + 2f_0 g_0' - f_0' g_0 = 0\,, \\ &3f_1''' + 2f_0 f_1'' - 3f_0' f_1' + 3f_0'' f_1 + 3 = 0\,. \end{aligned} \tag{3.4}$$

The pertinent boundary conditions are

$$\begin{aligned} &f_0(0) = 0\,, \qquad f_0''(0) = 0\,, \qquad f_0''(\infty) = \tfrac{1}{2}H''(0)\,, \\ &g_0'(0) = 0\,, \qquad g_0'(\infty) = G'(0)\,, \\ &f_1(0) = 0\,, \qquad f_1''(0) = 0\,. \end{aligned} \tag{3.5}$$

The third boundary condition for f_1 at $\eta \to \infty$ must be obtained from matching with the outer Goldstein solution, see equation (4.4).

Asymptotic expansions for $\eta \to \infty$ follow from (3.4):

$$f_0(\eta) \sim \tfrac{1}{4}H''(0)(\eta + A)^2\,, \qquad g_0(\eta) \sim G'(0)(\eta + A)\,, \tag{3.6}$$

where the constant A has been determined numerically as $A = 1.116283424$.

4. The outer Goldstein solution

This is the solution in the region where $\tilde{z} > O(x^{\frac{1}{3}})$. In this region, ξ and $\tilde{z}$ are the relevant coordinates. The equations of motion become

$$u\frac{\partial u}{\partial \xi} + 3\xi^2\tilde{w}\frac{\partial u}{\partial \tilde{z}} - \frac{3\xi^2}{1+\xi^3}v^2 = 3\xi^2\frac{\partial^2 u}{\partial \tilde{z}^2}, \qquad u\frac{\partial v}{\partial \xi} + 3\xi^2\tilde{w}\frac{\partial v}{\partial \tilde{z}} + \frac{3\xi^2}{1+\xi^3}uv = 3\xi^2\frac{\partial^2 v}{\partial \tilde{z}^2}. \tag{4.1}$$

Expansions for $\xi \to 0$ are

$$\tilde{\psi} = \tfrac{1}{2}H(\tilde{z}) + \xi J(\tilde{z}) + O(\xi^2), \quad u = \tfrac{1}{2}H'(\tilde{z}) + \xi J'(\tilde{z}) + O(\xi^2), \quad v = G(\tilde{z}) + \xi K(\tilde{z}) + O(\xi^2), \quad \tilde{w} = -\tfrac{1}{3}\xi^{-2}J(\tilde{z}) + O(\xi^{-1}). \tag{4.2}$$

Substitution of (4.2) into (4.1) leads to

$$H'J' - H''J = 0, \quad J(\tilde{z}) = CH'(\tilde{z}), \quad H'K - 2G'J = 0, \quad K(\tilde{z}) = 2CG'(\tilde{z}). \tag{4.3}$$

The constant C follows from matching with the inner Goldstein solution. Expansion of $\tilde{\psi}$ for $\tilde{z} \to 0$ yields

$$\tilde{\psi} = \tfrac{1}{2}\{\tfrac{1}{2}\tilde{z}^2H''(0) + \tfrac{1}{6}\tilde{z}^3H'''(0) + O(\tilde{z}^4)\} + C\xi\{\tilde{z}H''(0) + \tfrac{1}{2}\tilde{z}^2H'''(0) + O(\tilde{z}^3)\} + O(\xi^2).$$

Written in inner variables ξ, η this reads

$$\tilde{\psi} = \tfrac{1}{4}\xi^2\eta^2H''(0) + \tfrac{1}{12}\xi^3\eta^3H'''(0) + C\xi^2\eta H''(0) + \tfrac{1}{2}C\xi^3\eta^2H'''(0) + O(\xi^2),$$

where $\xi \to 0$, $\eta \to \infty$, but $\tilde{z} = \xi\eta \to 0$.

This last expression must agree with the asymptotic expansion (3.6) of the inner solution, i.e. $\tilde{\psi} \sim \frac{1}{4}H''(0)\xi^2(\eta + A)^2 + \xi^3 f_1(\eta)$, $\eta \to \infty$. The terms with $\xi^2\eta^2$ are in agreement. The terms with $\xi^2\eta$ lead to $C = \frac{1}{2}A$, while the most important term of $O(\xi^3)$ gives

$$\eta \to \infty, \quad f_1(\eta) \sim \tfrac{1}{12}\eta^3H'''(0),$$

and hence

$$f_1'''(\eta) \to \tfrac{1}{2}H'''(0).$$

It follows from (2.2) that $H'''(0) = -2\{G(0)\}^2 = -2$. So, the third boundary condition for f_1 is

$$f_1'''(\infty) = -1. \tag{4.4}$$

Finally, the outer solution for $\xi \to 0$ is

$$\tilde{\psi} = \tfrac{1}{2}H(\tilde{z}) + \tfrac{1}{2}A\xi H'(\tilde{z}) + O(\xi^2)\,, \qquad u = \tfrac{1}{2}H'(\tilde{z}) + \tfrac{1}{2}A\xi H''(\tilde{z}) + O(\xi^2)\,,$$
$$v = G(\tilde{z}) + A\xi G'(\tilde{z}) + O(\xi^2)\,, \qquad \tilde{w} = -\tfrac{1}{6}A\xi^{-2}H'(\tilde{z}) + O(\xi^{-1})\,. \tag{4.5}$$

There are two remarks to be made:

1. The terms denoted by the order symbols are not given by higher derivatives of the functions H and G, but are more complicated.
2. For $\xi \to 0$ the velocity $\tilde{w}$ becomes infinitely large as $O(\xi^{-2})$. This means that for very small values of ξ the boundary-layer equations are not valid. A double-deck structure of length $O(\mathrm{Re}^{-3/7})$ appears [3].

5. The asymptotic behaviour of the boundary layer for $r \to \infty$

For $r \to \infty$ the velocities will diminish while the boundary layer will become thicker. Hence

$$u \sim r^{\alpha} f'(\eta)\,, \qquad v \sim r^{\gamma} g(\eta)\,, \quad \text{where } \eta = \tilde{z}/r^{\beta} \tag{5.1}$$

and $\alpha < 0$, $\gamma < 0$, $\beta > 0$. The boundary conditions are

$$f(0) = 0\,, \qquad f'(\infty) = 0\,, \qquad f''(0) = 0\,, \qquad g'(0) = 0\,, \qquad g(\infty) = 0\,. \tag{5.2}$$

In the equation of continuity we have

$$\frac{\partial u}{\partial r} = \{\alpha f'(\eta) - \beta\eta f''(\eta)\} r^{\alpha-1}\,, \qquad \frac{u}{r} = f'(\eta) r^{\alpha-1}\,, \qquad \frac{\partial u}{\partial z} = f''(\eta) r^{\alpha-\beta}\,.$$

Thus,

$$\frac{\partial \tilde{w}}{\partial \tilde{z}} = \{-(\alpha+1) f'(\eta) + \beta\eta f''(\eta)\} r^{\alpha-1}\,.$$

Since

$$\frac{\partial \tilde{w}}{\partial \tilde{z}} = r^{-\beta} \frac{\partial \tilde{w}}{\partial \eta}\,,$$

we obtain

$$\frac{\partial \tilde{w}}{\partial \eta} = \{-(\alpha+1) f'(\eta) + \beta\eta f''(\eta)\} r^{\alpha+\beta-1}$$

and, after integration,

$$\tilde{w} = \{-(\alpha+\beta+1) f(\eta) + \beta\eta f'(\eta)\} r^{\alpha+\beta-1} + h(r)\,, \tag{5.3}$$

where $h(r) = 0$ since for $\eta = 0$, $f(\eta) = 0$ and $\tilde{w} = 0$.

We now substitute the assumptions (5.1) into the equations of motion (2.4). Then

$$\{\alpha f'^2 - (\alpha + \beta + 1)ff''\}r^{2\alpha - 1} - g^2 r^{2\gamma - 1} = f''' r^{\alpha - 2\beta} ,$$
$$\{(\gamma + 1)f'g - (\alpha + \beta + 1)fg'\}r^{\alpha + \gamma - 1} = g'' r^{\gamma - 2\beta} . \tag{5.4}$$

Since in the boundary layer the viscous term $g''r^{\gamma-2\beta}$ must play a role, this term must be of the same order as the other terms. Hence

$$\alpha + 2\beta = 1 . \tag{5.5}$$

In the first equation (5.4) this makes $r^{2\alpha-1}$ and $r^{\alpha-2\beta}$ of equal order, which implies that the remaining term containing $r^{2\gamma-1}$ cannot be of larger order. Thus

$$\gamma \leqslant \alpha . \tag{5.6}$$

We shall now investigate the two cases $\gamma < \alpha$ and $\gamma = \alpha$. We begin with $\gamma < \alpha$. Then, for $r \to \infty$, the first equation (5.4) simplifies to

$$\alpha f'^2 - (\alpha + \beta + 1)ff'' = f''' .$$

Eliminating α with the aid of (5.5), we obtain

$$(1 - 2\beta)f'^2 - (2 - \beta)ff'' = f''' ,$$

which can also be written as

$$(3 - 3\beta)f'^2 - (2 - \beta)(f'^2 + ff'') = f''' .$$

This equation can be integrated as

$$(3 - 3\beta)\int_0^\infty f'^2 \, d\eta - (2 - \beta)ff'\Big|_0^\infty = f''\Big|_0^\infty .$$

Since all terms except the first one vanish due to the boundary conditions, the first term must also be zero. This means $\beta = 1$ and from (5.5) it follows that $\alpha = -1$. The differential equation then becomes

$$f''' + ff'' + f'^2 = 0$$

and, integrated,

$$f'' + ff' = C ,$$

where $C = 0$ due to $f(0) = 0$ and $f''(0) = 0$.

A second integration yields

$$f' + \tfrac{1}{2}f^2 = C_1 ,$$

where

$$C_1 = \tfrac{1}{2}\{f(\infty)\}^2 = f'(0) . \tag{5.7}$$

The last equation can be solved with the result that

$$f(\eta) = C \tanh \tfrac{1}{2}\eta C , \quad \text{where } C = f(\infty) . \tag{5.8}$$

After substitution of the values for α and β, the second equation (5.4) becomes

$$(\gamma + 1)f'g - fg' = g'' , \qquad g'(0) = 0 , \qquad g(\infty) = 0 , \tag{5.9}$$

which is an eigenvalue problem.

The independent variable η will be replaced by a new variable

$$t = \tanh \tfrac{1}{2}\eta C = C^{-1}f(\eta) .$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\eta} = C^{-1}f' \frac{\mathrm{d}}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} = C^{-2}f'^2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + C^{-1}f'' \frac{\mathrm{d}}{\mathrm{d}t} .$$

Substitution into (5.9) yields

$$f' \frac{\partial^2 g}{\partial t^2} = C^2(\gamma + 1)g .$$

Using the solution of f, given by (5.8), we obtain the result

$$(1 - t^2) \frac{\partial^2 g}{\partial t^2} - 2(\gamma + 1)g = 0 , \quad t \in [0, 1] . \tag{5.10}$$

It follows from [7, §22.6, Eq. (22.6.1)] that equation (5.10) has only a bounded solution if

$$-2(\gamma + 1) = n(n - 1) , \quad n = 0, 1, \ldots .$$

From $n = 2$ onward the bounded solutions are Jacobian polynomials, orthogonal on the interval $[-1, 1]$ with weight function $(1 - t^2)^{-1}$. All these polynomials contain a factor $1 - t^2$ and hence satisfy the boundary condition $g = 0$ at $t = 1$ ($\eta = \infty$). Only those corresponding to even values of n satisfy also the boundary conditions $g' = 0$ at $t = 0$ ($\eta = 0$). For $n = 2$, we obtain $\gamma = -2$ and the solution of (5.10) is $g(t) = C_2(1 - t^2)$ or

$$g(\eta) = C_2 \operatorname{sech}^2 \tfrac{1}{2}\eta C . \tag{5.11}$$

The smaller values of γ corresponding to $n = 4, 6, \ldots$ may correspond to further terms in the asymptotic expansion of v.

Finally, we have to show that the case $\gamma = \alpha$ does not lead to a solution. The equations

(5.4) then are

$$\alpha f'^2 - (\alpha + \beta + 1)ff'' - g^2 = f''' , \qquad (\alpha + 1)f'g - (\alpha + \beta + 1)fg' = g'' . \tag{5.12}$$

By elimination of α with the aid of (5.5), the first equation can be written in the form

$$(3 - 3\beta)f'^2 - (2 - \beta)(f'^2 + ff'') - g^2 = f''' .$$

Integrating this equation between $\eta = 0$ and $\eta = \infty$ and taking into account the boundary conditions, the result is

$$(3 - 3\beta) \int_0^\infty f'^2 \, d\eta - \int_0^\infty g^2 \, d\eta = 0 ,$$

which requires $\beta < 1$.

The same procedure applied to the second equation (5.12) leads to

$$(4 - 3\beta)f'g - (2 - \beta)(f'g + fg') = g''$$

and hence

$$(4 - 3\beta) \int_0^\infty f'g \, d\eta = 0 .$$

Since f' and g are functions which do not change sign, the only possibility to satisfy the last equation is $\beta = 4/3$ and this is in contradiction to the previously found requirement $\beta < 1$.

This completes the investigation of the asymptotic behaviour with the conclusion that for

$$r \to \infty , \quad u \sim r^{-1} f'(\eta) , \quad v \sim r^{-2} g(\eta) , \quad \tilde{w} \sim r^{-1}\{-f(\eta) + \eta f'(\eta)\} , \quad \eta = \tilde{z}/r ,$$

$$f(\eta) = C \tanh \tfrac{1}{2}\eta C , \quad f'(\eta) = C_1 \operatorname{sech}^2 \tfrac{1}{2}\eta C , \quad g(\eta) = C_2 \operatorname{sech}^2 \tfrac{1}{2}\eta C .$$

These results, as well as the relation (5.7), are fully confirmed by the numerical computations described in Section 6. The constants appear to be

$$C = f(\infty) = 0.79 , \quad C_1 = \tfrac{1}{2}C^2 = f'(0) = 0.31 , \quad C_2 = g(0) = 0.29 .$$

6. The numerical computations

We have to solve the set of equations (2.4) with pertinent boundary conditions. After elimination of $\tilde{w}$, this set can be written as

$$\frac{1}{r}\frac{\partial \tilde{\psi}}{\partial \tilde{z}} = u , \quad \frac{1}{r}\frac{\partial(u, \tilde{\psi})}{\partial(r, \tilde{z})} - \frac{v^2}{r} = \frac{\partial^2 u}{\partial \tilde{z}^2} , \quad \frac{1}{r}\frac{\partial(v, \tilde{\psi})}{\partial(r, \tilde{z})} + \frac{uv}{r} = \frac{\partial^2 v}{\partial \tilde{z}^2} . \tag{6.1}$$

In order to obtain an accurate solution near the singular point $r = 1$, $\tilde{z} = 0$, it is necessary to distinguish three regions of integration (see Fig. 1). Region I is the region of the inner Goldstein solution and region II that of the outer Goldstein solution. The coordinates in

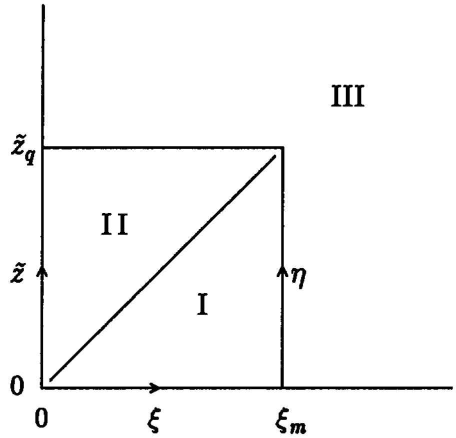

Fig. 1. The three regions of integration.

region I are (ξ, η), where

$$\xi = x^{1/3} = (r-1)^{1/3} , \qquad \eta = \tilde{z}/\xi . \tag{6.2}$$

The boundary between regions I and II is determined by $\eta = \eta_q$, where η_q is a value beyond which the inner solution is described with sufficient accuracy by its asymptotic behavior (exponentially decreasing deviation). We also introduce a value ξ_m beyond which no distinction needs to be made between the inner and the outer solution. The regions then are defined as follows

region I: $0 < \xi \leq \xi_m , \quad 0 \leq \eta < \eta_q ,$

region II: $0 < \xi \leq \xi_m , \quad \xi\eta_q \leq \tilde{z} < \tilde{z}_q = \xi_m \eta_q ,$

region III: $0 < \xi \leq \xi_m , \quad \tilde{z}_q \leq \tilde{z}$ and $\xi_m < \xi , \quad 0 \leq \tilde{z} .$

The equations are solved by a difference method. Since the boundary-layer variables show the quickest change near $\tilde{z} = 0$, we take more points near the plane $\tilde{z} = 0$ by introducing in region III

$$\tilde{z} = \tilde{z}_{\max} \frac{\sinh \beta\mu}{\sinh \beta} \quad \text{with } \tilde{z}_{\max} = 25 , \quad \beta = 5 , \quad 0 \leq \mu \leq 1 .$$

In μ-direction we will take p equidistant points defined by $\mu_j = jh$ with $j = 0, 1, \ldots, p$ and $h = 1/p$. The boundary $\tilde{z} = \tilde{z}_q$ between the regions II and III is given by $j = q$. The points in region I are determined by $\xi_i, \eta_l (1 \leq i \leq m, 0 \leq l \leq q)$. The ξ_i are defined below and $\eta_l = \tilde{z}_l / \xi_m$. In region II the points are given by ξ_i, y_k where $1 \leq i \leq m$ and $i \leq k \leq m$ (Fig. 2). These coordinates are equal to

$$y_k = y_m \, e^{(k-m)\alpha} , \qquad \xi_i = \xi_m \, e^{(i-m)\alpha} \quad \text{with } \alpha = \ln(\tilde{z}_q / \tilde{z}_{q-1}) .$$

This has the advantage that also at the boundaries from region I to II and from region II to

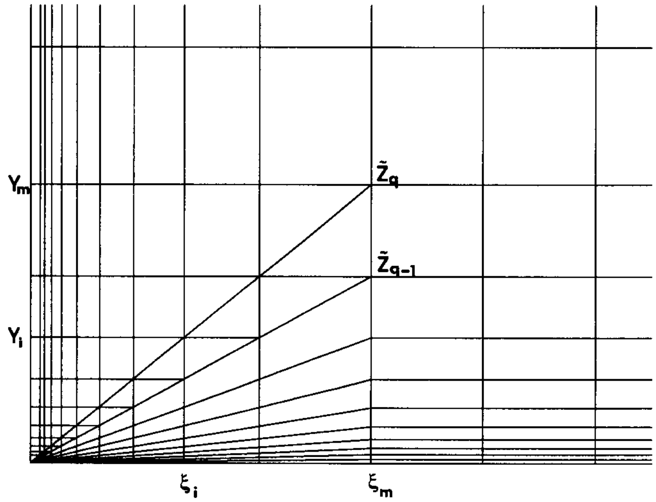

Fig. 2. Definition of coordinates ξ_i, y_i.

III, we can use steps of equal size. Indeed

$$\frac{y_{m-1}}{y_m} = e^{-\alpha} = \frac{\tilde{z}_{q-1}}{\tilde{z}_q} = \frac{\sinh \beta\mu_{q-1}}{\sinh \beta\mu_q}$$

and, therefore, y_{m-1} still fits into the μ-division of region III. This means that we can take the boundary between regions II and III as part of region III and we retain equidistant steps in μ.

In region II we introduce $\nu_k = kh$, again with $h = 1/p$ and we take ν as continuous variable with equidistant points ν_k. Then

$$y_k = y_0 \, e^{\nu_k \alpha/h} \quad \text{or} \quad y = y_0 \, e^{\nu\alpha/h} .$$

Since

$$\frac{e^{\nu_{k-1}\alpha/h}}{e^{\nu_k \alpha/h}} = \frac{y_{k-1}}{y_k} = e^{-\alpha} = \frac{\tilde{z}_{q-1}}{\tilde{z}_q} = \frac{\eta_{q-1}}{\eta_q} = \frac{\sinh \beta\mu_{q-1}}{\sinh \beta\mu_q} ,$$

which shows that the points (ξ_i, y_{i-1}) and (ξ_i, y_i) are characterized both by ν_{i-1}, ν_i and by μ_{q-1}, μ_q. Therefore, the boundary between regions I and II will be taken as part of region II and we retain equal steps in the ν-variable.

We now transform equations (6.1) to the variables ξ, μ of region I. The Jacobian is

$$\frac{\partial(u, \tilde{\psi})}{\partial(r, \tilde{z})} = \frac{\partial(u, \tilde{\psi})}{\partial(\xi, \mu)} \frac{\partial(\xi, \mu)}{\partial(r, \tilde{z})} = \frac{\partial(u, \tilde{\psi})}{\partial(\xi, \mu)} \frac{\partial \mu}{\partial \tilde{z}} \Big/ (3\xi^2)$$

with

$$\tilde{z} = \frac{\xi}{\xi_m} \tilde{z}_{\max} \frac{\sinh \beta\mu}{\sinh \beta} \quad \text{in region I} . \tag{6.3}$$

The set of equations then becomes

$$\frac{\partial \tilde{\psi}}{\partial \mu} = ru\left(\frac{\partial \mu}{\partial \tilde{z}}\right)^{-1},$$

$$u\,\frac{\partial u}{\partial \xi} = \frac{1}{r}\,\frac{\partial \tilde{\psi}}{\partial \xi}\,\frac{\partial u}{\partial \mu}\,\frac{\partial \mu}{\partial \tilde{z}} + 3\xi^2\left\{\frac{\partial^2 u}{\partial \mu^2}\left(\frac{\partial \mu}{\partial \tilde{z}}\right)^2 + \frac{\partial u}{\partial \mu}\,\frac{\partial^2 \mu}{\partial \tilde{z}^2} + \frac{v^2}{r}\right\}, \qquad (6.4)$$

$$u\,\frac{\partial v}{\partial \xi} = \frac{1}{r}\,\frac{\partial \tilde{\psi}}{\partial \xi}\,\frac{\partial v}{\partial \mu}\,\frac{\partial \mu}{\partial \tilde{z}} + 3\xi^2\left\{\frac{\partial^2 v}{\partial \mu^2}\left(\frac{\partial \mu}{\partial \tilde{z}}\right)^2 + \frac{\partial v}{\partial \mu}\,\frac{\partial^2 \mu}{\partial \tilde{z}^2} - \frac{uv}{r}\right\},$$

In region II the set of equations is the same but with μ replaced by ν and $\tilde{z}$ by y. In region III we have the same set of equations as in region I, the only difference being the relation between $\tilde{z}$ and μ, which is

$$\tilde{z} = \tilde{z}_{\max}\,\frac{\sinh \beta\mu}{\sinh \beta} \quad \text{in region III}.$$

The boundary conditions at $\tilde{z} = 0$, where we have $\partial\mu/\partial\tilde{z} \neq 0$, are

$$\tilde{z} = 0: \quad \tilde{\psi} = 0\,, \quad \frac{\partial u}{\partial \mu} = 0\,, \quad \frac{\partial v}{\partial \mu} = 0\,,$$

$$\tilde{z} = \tilde{z}_{\max}: \quad u = 0\,, \quad v = 0\,.$$

For solving the equations, we use the Crank–Nicolson method combined with an iterative Newton–Raphson procedure due to the non-linearities. As starting values at $\xi = 0$ the values obtained in Section 2 are used. The inner and outer solutions from Sections 3 and 4 were taken as initial values at $\xi = \xi_1$ for the Newton–Raphson procedure. Initial values at $\xi = \xi_2, \ldots, \xi_m$ were obtained from extrapolation in ξ-direction.

The calculations have been performed with

$$\xi_m = 0.1\,, \quad m = 30\,, \quad q = 10\,, \quad p = 50$$

and also with

$$\xi_m = 0.1\,, \quad m = 60\,, \quad q = 20\,, \quad p = 100\,,$$

in both cases until $\xi = 1$. From $\xi_m = 0.1$ onwards $\Delta\xi$ has been taken equal to 0.0125. Some results obtained with the finer mesh are presented in Table 1.

Since the suspicion existed that the influence of the singularity at the edge of the disk was not too important, calculations have also been performed with the equations of region III used in the whole quarter plane. Here also $p = 100$ and $\Delta\xi = 0.0125$. Results are given in Table 2. For extremely small values of ξ and $\tilde{z}$ an irregularity occurs when the steps are taken increasingly smaller. But as soon as ξ becomes larger, the values are again reliable. The larger the ξ, the smaller the difference between the calculations with and without regions I and II. This is due to the smoothing character of solutions of parabolic equations. From a comparision of Tables 1 and 2 it is seen that unless one wants the solution for extremely small values of ξ (and these can be obtained analytically as inner and outer solutions), it is superfluous to introduce regions I and II.

Table 1. Results for u, v and $\tilde{w}$ obtained with regions I and II

$\tilde{z}$	u: $\xi = 0.0$	0.1	0.2	0.4	0.6	0.8	1.0
25.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15.1621	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007
9.1943	0.0003	0.0002	0.0002	0.0002	0.0003	0.0022	0.0103
5.5734	0.0066	0.0060	0.0054	0.0048	0.0070	0.0215	0.0448
3.3751	0.0431	0.0394	0.0360	0.0321	0.0433	0.0747	0.0929
2.0384	0.1160	0.1081	0.1005	0.0930	0.1143	0.1355	0.1272
1.2219	0.1727	0.1668	0.1604	0.1597	0.1796	0.1740	0.1441
0.7174	0.1752	0.1799	0.1824	0.2044	0.2163	0.1917	0.1510
0.3959	0.1355	0.1541	0.1757	0.2264	0.2324	0.1988	0.1536
0.1756	0.0752	0.1135	0.1656	0.2351	0.2384	0.2014	0.1546
0.0000	0.0000	0.0917	0.1620	0.2374	0.2399	0.2020	0.1548

$\tilde{z}$	v: $\xi = 0.0$	0.1	0.2	0.4	0.6	0.8	1.0
25.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15.1621	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
9.1943	0.0004	0.0003	0.0003	0.0002	0.0003	0.0017	0.0055
5.5734	0.0087	0.0079	0.0071	0.0063	0.0080	0.0170	0.0238
3.3751	0.0607	0.0550	0.0499	0.0440	0.0506	0.0591	0.0495
2.0384	0.1967	0.1785	0.1621	0.1430	0.1366	0.1075	0.0678
1.2219	0.3964	0.3610	0.3288	0.2826	0.2183	0.1381	0.0767
0.7174	0.5984	0.5479	0.5012	0.3988	0.2655	0.1523	0.0804
0.3959	0.7644	0.7065	0.6397	0.4626	0.2863	0.1580	0.0818
0.1756	0.8927	0.8229	0.7195	0.4892	0.2942	0.1601	0.0823
0.0000	1.0000	0.8733	0.7436	0.4961	0.2962	0.1606	0.0824

$\tilde{z}$	$\tilde{w}$: $\xi = 0.0$	0.1	0.2	0.4	0.6	0.8	1.0
25.0000	$-\infty$	−0.8833	−0.8825	−0.8845	−0.8608	−0.6390′	−0.4405
15.1621	$-\infty$	−0.8833	−0.8826	−0.8845	−0.8608	−0.6383	−0.4331
9.1943	$-\infty$	−0.8927	−0.8844	−0.8844	−0.8576	−0.6164	−0.3681
5.5734	$-\infty$	−1.1061	−0.9249	−0.8822	−0.8092	−0.4909	−0.2179
3.3751	$-\infty$	−2.3347	−1.1538	−0.8610	−0.6198	−0.2682	−0.0855
2.0384	$-\infty$	−4.8282	−1.6155	−0.7764	−0.3491	−0.1043	−0.0260
1.2219	$-\infty$	−6.8902	−1.9881	−0.6001	−0.1633	−0.0360	−0.0075
0.7174	$-\infty$	−7.2249	−2.0111	−0.3949	−0.0764	−0.0134	−0.0024
0.3959	$-\infty$	−6.1454	−1.6044	−0.2272	−0.0371	−0.0057	−0.0009
0.1756	$-\infty$	−3.9859	−0.8528	−0.1022	−0.0156	−0.0022	−0.0003
0.0000		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2. Results for u, v and $\tilde{w}$ obtained without separate regions I and II

	u						
$\tilde{z}$	$\xi = 0.0$	0.1	0.2	0.4	0.6	0.8	1.0
25.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15.1621	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007
9.1943	0.0003	0.0002	0.0002	0.0002	0.0003	0.0022	0.0103
5.5734	0.0066	0.0060	0.0054	0.0048	0.0070	0.0215	0.0448
3.3751	0.0431	0.0393	0.0359	0.0321	0.0433	0.0747	0.0929
2.0384	0.1160	0.1079	0.1005	0.0930	0.1144	0.1355	0.1272
1.2219	0.1727	0.1667	0.1603	0.1597	0.1795	0.1739	0.1440
0.7174	0.1752	0.1799	0.1824	0.2044	0.2163	0.1917	0.1510
0.3959	0.1355	0.1553	0.1759	0.2264	0.2323	0.1988	0.1536
0.1756	0.0752	0.1141	0.1659	0.2351	0.2383	0.2013	0.1545
0.0000	0.0000	0.0922	0.1623	0.2373	0.2399	0.2020	0.1548

	v						
$\tilde{z}$	$\xi = 0.0$	0.1	0.2	0.4	0.6	0.8	1.0
25.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15.1621	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004
9.1943	0.0004	0.0003	0.0003	0.0002	0.0004	0.0017	0.0055
5.5734	0.0087	0.0079	0.0071	0.0063	0.0081	0.0170	0.0238
3.3751	0.0607	0.0549	0.0499	0.0440	0.0506	0.0591	0.0494
2.0384	0.1967	0.1782	0.1620	0.1430	0.1365	0.1074	0.0677
1.2219	0.3964	0.3604	0.3286	0.2825	0.2182	0.1381	0.0767
0.7174	0.5984	0.5470	0.5008	0.3985	0.2654	0.1523	0.0804
0.3959	0.7644	0.7028	0.6387	0.4622	0.2861	0.1579	0.0818
0.1756	0.8927	0.8214	0.7183	0.4888	0.2940	0.1600	0.0823
0.0000	1.0000	0.8724	0.7423	0.4956	0.2960	0.1605	0.0824

	$\tilde{w}$						
$\tilde{z}$	$\xi = 0.0$	0.1	0.2	0.4	0.6	0.8	1.0
25.0000	$-\infty$	−0.8833	−0.8826	−0.8845	−0.8606	−0.6389	−0.4404
15.1621	$-\infty$	−0.8833	−0.8826	−0.8845	−0.8606	−0.6382	−0.4330
9.1943	$-\infty$	−0.8926	−0.8844	−0.8844	−0.8574	−0.6162	−0.3680
5.5734	$-\infty$	−1.1043	−0.9245	−0.8822	−0.8089	−0.4907	−0.2178
3.3751	$-\infty$	−2.3225	−1.1511	−0.8607	−0.6194	−0.2681	−0.0855
2.0384	$-\infty$	−4.7956	−1.6080	−0.7758	−0.3488	−0.1042	−0.0260
1.2219	$-\infty$	−6.8424	−1.9764	−0.5992	−0.1631	−0.0359	−0.0075
0.7174	$-\infty$	−7.1775	−1.9977	−0.3942	−0.0764	−0.0134	−0.0024
0.3959	$-\infty$	−6.1097	−1.5937	−0.2268	−0.0371	−0.0057	−0.0009
0.1756	$-\infty$	−3.9912	−0.8476	−0.1020	−0.0156	−0.0022	−0.0003
0.0000		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 3. Results for u, v and $\tilde{w}$ for large values of r

	u						
$\tilde{z}/r$	$r=1$	2	4	8	16	32	64
25.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15.1621	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9.1943	0.0003	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000
5.5734	0.0066	0.0044	0.0033	0.0018	0.0009	0.0005	0.0002
3.3751	0.0431	0.0284	0.0178	0.0093	0.0047	0.0024	0.0012
2.0384	0.1160	0.0753	0.0420	0.0215	0.0108	0.0054	0.0027
1.2219	0.1727	0.1172	0.0615	0.0310	0.0155	0.0078	0.0039
0.7174	0.1752	0.1402	0.0716	0.0359	0.0180	0.0090	0.0045
0.3959	0.1355	0.1501	0.0758	0.0380	0.0190	0.0095	0.0047
0.1756	0.0752	0.1538	0.0774	0.0387	0.0194	0.0097	0.0048
0.0000	0.0000	0.1548	0.0778	0.0389	0.0195	0.0097	0.0049

	v						
$\tilde{z}/r$	$r=1$	2	4	8	16	32	64
25.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
15.1621	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9.1943	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
5.5734	0.0087	0.0023	0.0008	0.0002	0.0001	0.0000	0.0000
3.3751	0.0607	0.0151	0.0043	0.0011	0.0003	0.0001	0.0000
2.0384	0.1967	0.0401	0.0101	0.0025	0.0006	0.0002	0.0000
1.2219	0.3964	0.0624	0.0149	0.0037	0.0009	0.0002	0.0001
0.7174	0.5984	0.0747	0.0173	0.0042	0.0011	0.0003	0.0001
0.3959	0.7644	0.0799	0.0183	0.0045	0.0011	0.0003	0.0001
0.1756	0.8927	0.0819	0.0187	0.0046	0.0011	0.0003	0.0001
0.0000	1.0000	0.0824	0.0188	0.0046	0.0012	0.0003	0.0001

	$\tilde{w}$						
$\tilde{z}/r$	$r=1$	2	4	8	16	32	64
25.0000	$-\infty$	−0.4414	−0.2028	−0.0993	−0.0494	−0.0247	−0.0124
15.1621	$-\infty$	−0.4414	−0.2028	−0.0993	−0.0494	−0.0247	−0.0124
9.1943	$-\infty$	−0.4319	−0.2009	−0.0982	−0.0489	−0.0244	−0.0122
5.5734	$-\infty$	−0.4046	−0.1790	−0.0867	−0.0431	−0.0215	−0.0108
3.3751	$-\infty$	−0.2804	−0.1151	−0.0547	−0.0271	−0.0135	−0.0068
2.0384	$-\infty$	−0.1267	−0.0476	−0.0222	−0.0109	−0.0054	−0.0027
1.2219	$-\infty$	−0.0406	−0.0139	−0.0064	−0.0031	−0.0016	−0.0008
0.7174	$-\infty$	−0.0110	−0.0033	−0.0015	−0.0007	−0.0004	−0.0002
0.3959	$-\infty$	−0.0029	−0.0006	−0.0003	−0.0001	−0.0001	−0.0000
0.1756	$-\infty$	−0.0008	−0.0001	−0.0000	−0.0000	−0.0000	−0.0000
0.0000		−0.0000	−0.0000	−0.0000	−0.0000	−0.0000	−0.0000

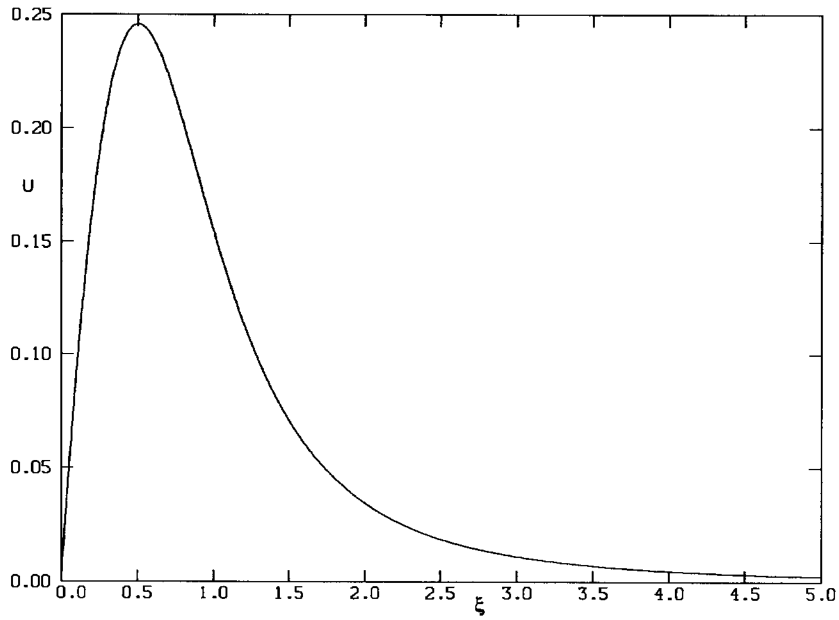

Fig. 3. The radial velocity u for $\tilde{z} = 0$.

Using this conclusion, a final calculation has been made up to arbitrarily large values of ξ. Taking into account the linearly increasing thickness of the boundary layer (Section 5), the relation between $\tilde{z}$ and μ has now been taken as

$$\tilde{z} = \tilde{z}_{max} r \frac{\sinh \beta\mu}{\sinh \beta}\,, \quad \text{again with } \tilde{z}_{max} = 25 \text{ and } \beta = 5\,.$$

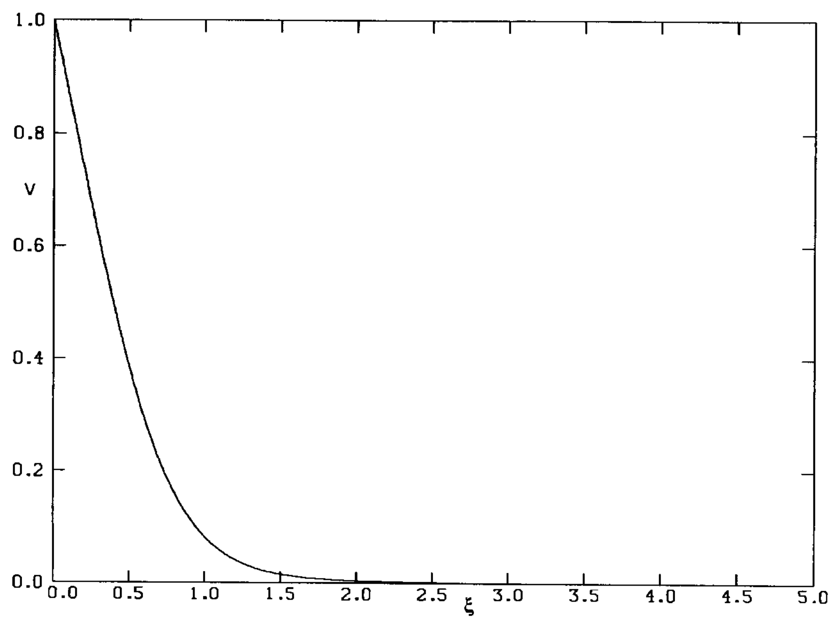

Fig. 4. The tangential velocity v for $\tilde{z} = 0$.

Beyond $r = 2$, $\xi = 1$, a new variable in r-direction has been used, viz. $s = 1 - 2/r$, which brings $r \to \infty$ to $s = 1$. Then for $r \to \infty$, u diminishes like $1 - s$ and v like $(1 - s)^2$. Results for u, v and $\tilde{w}$ are presented in Table 3. All conclusions derived analytically in Section 5 are confirmed by the computations.

Finally, Figs. 3 and 4 show the functions $u(\xi, 0)$ and $v(\xi, 0)$ outside the disk.

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